Oceanic coastline and super-universality of percolation clusters.

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New fractal subset of a rough surface, the "oceanic coastline", is defined. For random Gaussian surfaces with negative Hurst exponent $H < 0$, "oceanic coastlines" are mapped to the percolation clusters of the (correlated) percolation problem. In the case of rough self-affine surfaces ($H \geq 0$), the fractal dimension of the "oceanic coastline" d_c is calculated numerically as a function of the roughness exponent H (using a novel technique of minimizing finite-size effects). For $H = 0$, the result $d_c \approx 1.896$ coincides with the analytic value for the percolation problem (91/48), suggesting a super-universality of d_c for correlated percolation problem.

Random surfaces are common objects which can be found in various situations; good examples are deposited metal films [1], fractured surfaces [2, 3], streamfunctions of two-dimensional (2D) turbulent flows [4] etc. For various applications, it is important to know the statistical geometrical properties (statistical topography) of such surfaces. This problem has been studied with a considerable success, both numerically and analytically [1, 2, 3, 4, 5, 6, 7].

Here we introduce a new scale-invariant measure of such surfaces, the fractal dimension of the "oceanic coastlines". An "oceanic coastline" includes the outer perimeter of the ocean (the "continental coastline"), and the coastline of "oceanic islands", i.e. of such islands, which are adjacent to the largest (possibly infinite) connected flooded region, when water floods the surface up to a certain level; "oceanic islands" are to be opposed to such islands, which are inside the lakes. Thus, each flooding level defines its own "oceanic coastline". Here we show that for the so called "monoscale" random surfaces (with bounded height, cf. [5]), the "oceanic coastlines" can be mapped to the percolation clusters of the correlated percolation problem. For rough surfaces (with unbounded height), the fractal dimension d_c is calculated numerically, using a novel technique of minimizing the finite-size effects. The results suggest that for the correlated percolation (and hence for the "monoscale" surfaces), there is a super-universality, $d_c \equiv 91/48$.

An important characteristic of random surfaces [given by the surface height $\psi(x, y) \equiv \psi(\boldsymbol{r})$ over the two-dimensional plane] is the Hurst exponent H; in the case of Gaussian surfaces, all the statistical topography scaling exponents are functions of H. The most intuitive definition of the Hurst exponent is given by the scaling law of the surface height drop at a distance $a \equiv |\boldsymbol{a}|$ (c.f. [8]):

$$\langle (\psi(\boldsymbol{r}) - \psi(\boldsymbol{r} + \boldsymbol{a}))^2 \rangle \propto |\boldsymbol{a}|^{2H}. \qquad (1)$$

Here angular braces denote averaging over an ensemble (or over the radius vector $\boldsymbol{r}$); we assume that a is larger than the lower cut-off scale a_0, and $0 \leq H \leq 1$. More general definition (not bounded to $0 \leq H \leq 1$) of the exponent H is given via the power spectrum $P_{\boldsymbol{k}}$ [with $\langle \psi_{\boldsymbol{k}} \psi_{\boldsymbol{k}'} \rangle = P_{\boldsymbol{k}} \delta(\boldsymbol{k} + \boldsymbol{k}')$, where $\psi_{\boldsymbol{k}}$ is the Fourier transform of the surface height; c.f. [4]]:

$$P_{\boldsymbol{k}} \propto |\boldsymbol{k}|^{-2H-2} \quad \text{for} \quad |\boldsymbol{k}| \ll a_0^{-1}. \qquad (2)$$

According to the value of H, surfaces can be divided into three classes [5]:
(i) $H < 0$: "monoscale", surface looks like a landscape with hills of approximately same size, the surface height is bounded;
(ii) $0 \leq H < 1$: "rough", looks like rocky mountains with smaller and higher peaks and unbounded height;
(iii) $H \geq 1$: "smooth", the largest scale defines the slope of the surface.

For "monoscale" surfaces, there is a simple mapping between statistical topography and percolation problem, developed by Ziman and Weinrib [9, 10] [the minima of the surface define the sites of an irregular percolation lattice; the saddle-points S_{ij} (between i-th and j-th minima) correspond to the bonds of the lattice (which can be drawn as the steepest descend paths from S_{ij} to the i-th and j-th minima); when the surface is flooded by water, a bond is declared broken if the corresponding saddle is ("dry"), i.e. above the water level; the water level itself is the counterpart of the bond-keeping probability p]. This mapping can be also applied to the case of rough surfaces. Then, however, the lattice problem is no more the (correlated) percolation problem. In particular, there is no critical probability $p = p_c$ at which the global connectivity disappears. Indeed if we "look far enough", we can find as deep (hence also as wide) connected wet region as we want, because the height is unbounded. Therefore, the mapping between the percolation problem and statistical topography allows us to extend various concepts of the percolation problem outside the range of applicability of the very percolation problem.

A straightforward example here is the hull of a percolation cluster: this is defined as the outer perimeter of a percolation cluster. Since the percolation cluster corresponds to a connected wet region, its outer perimeter corresponds to an isoline of the surface. It has been shown that for uncorrelated percolation, its fractal dimension $d_h = 7/4$ [11]. Note that percolation with weak correlation (with $H < -3/4$) belongs to the same universality class as the uncorrelated percolation [12]. Meanwhile, for

rough surfaces, $d_h \approx 1.5 - 0.5H$ [13], with an exact result $d_h = 1.5$ for $H = 0$ [6]. Therefore, for $-3/4 < H < 0$, one can expect a non-trivial decreasing function $d_h(H)$, with $d_h(-3/4) = 1.75$ and $d_h(0) = 1.5$. Further, for the percolation problem, one can distinguish between the hull and the unscreened perimeter. This is because the hull is very twisted, and typically, there are narrow "necks" of peninsulas separating "gulfs". We can postulate that these "necks" are too narrow, so that the coastline of the respective peninsulas is "screened"; the rest of the coastline is called "unscreened perimeter". For percolation clusters, the fractal dimension of the unscreened perimeter is $d_u = 4/3$ [11]. Since for rough surfaces, narrow "necks" are untypical, $d_u(H) = d_h(H)$ for $H \geq 0$. Comparing $d_u(-3/4) = 4/3$ and $d_u(0) = 1.5$, one can expect an increasing function $d_u(H)$ for $-3/4 < H < 0$.

Now we are ready to turn to the percolation cluster. Each bond of the cluster is touched by the perimeter (hull) from both sides; thus, the length of the net perimeter scales as the cluster size. However, the perimeter touching it can be either external or internal, see Fig. 1. The sum of external and internal perimeters is the sum of *(a)* the coastline between the "ocean" (=cluster) and the "continent" (=infinite cluster of broken bonds), and *(b)* the coastline between the "ocean" and the "islands" (=broken bond clusters inside the cluster of undamaged bonds); we call this the "oceanic coastline". Note that the coastlines of the "lakes" inside the "islands", "islands" inside the "lakes" inside the "islands", etc., do not touch the "ocean", i.e. the bonds of the cluster, and are not relevant here (including these coastlines, too, would result in the set of all the isolines at the given height, the fractal dimension of which $d_f = 2 - H$ for $0 \leq H \leq 1$; for $H \leq 0$, there is a super-universality, $d_f \equiv 2$).

In order to calculate numerically the fractal dimension of the "oceanic coastlines", we have opted for the following procedure. First, 1+1-dimensional [(1+1)D] random surfaces (which are assumed to belong to the same universality class as the statistically isotropic 2D surfaces) were generated on the lattice of the four-vertex model [13]. The one-dimensional discretized fractional Brownian functions $f_H(i)$ were generated as fractional integrals of uncorrelated random sequence of "spins" $s_i = \pm 1$ as follows. The aim function $g_H(i)$ is found as

$$g_H(i) = \sum_{j=0}^{j} s_j |i - j|^{H-0.5}. \tag{3}$$

Further, the discretized function $f_H(i)$ is found incrementally, starting from $H = 0$, by rounding $g_H(i)$ to such a nearest integer that

$$f_H(i) - f_H(i-1) = \pm 1. \tag{4}$$

Finally, the (1+1)D function is given by the superposition of two 1D-functions,

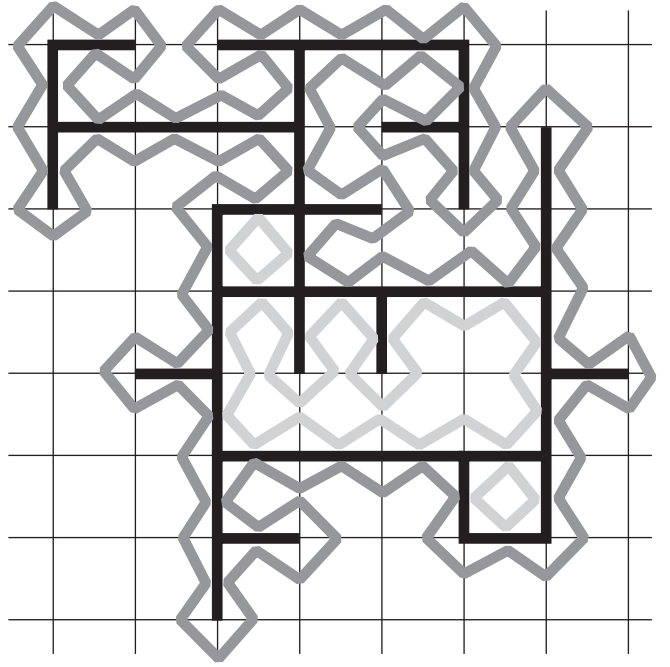$$F_H(i, j) = f_{H1}(i) - f_{H2}(j). \tag{5}$$

FIG. 1: All the bonds of a percolation cluster (black bold lines) are touched from both sides, either by external perimeter (hull, dark grey lines) or internal perimeter (light grey lines).

After generating a 2D surface on a square polygon of side length N, we started "flooding" the polygon. To this end, at the perimeter of the polygon, the lowest point P_0 was found. Further, "water" was incrementally "injected" into that point, until the flooded region connected opposite edges of the polygon (either left and right or top and bottom), see Fig. 2. Such a procedure guarantees that only the "ocean" (and not the "lakes") becomes wet. At that stage, three quantities were recorded: L_1 — the coastline length, L_2 — the number of cells touching the coastline, and L_3 — the coastline length immediately before achieving the critical flood level. The procedure was repeated (for each polygon size and Hurst exponent value) 10^8 or more times. The covered polygon sizes are given by formulae $N = 128 \cdot 2^k$ and $N = 196 \cdot 2^k$, with $k = 1, 2, 3, 4$, and $N = 4096$ (the largest polygon size was used only for $H = 0$). The simulations were performed on the cluster of ten AthlonXP-1700 workstations during three months, using an assembly-optimized code.

It was expected that the finite-size scaling of the average values $l_i(N) = \langle L_i(N) \rangle$ allows us to estimate the fractal dimension d_c. Such an asymptotic behavior was observed, indeed, for larger Hurst exponents ($H \geq 0.375$). However, for smaller values of H (and in particular, for $H = 0$), the convergence $l_i(N) \rightarrow A N^{d_c}$ was very slow, see Fig. 3 (here A denotes a constant). Consequently, if we assume that l_i can be expanded asymptotically,

$$l_\nu = \sum_{\nu=0}^{\infty} A_{\nu i} N^{\alpha_{\nu i}}, \alpha_{(\nu+1)i} < \alpha_{\nu i}, \tag{6}$$

the non-leading exponents ($\nu \neq 0$) have to be very close to the leading term. In order to resolve this difficulty, we made an assumption that for the three recorded quantities l_i ($i = 1, 2, 3$), the three leading exponents ($\nu = 0, 1, 2$) are the same, $\alpha_{\nu i} = \alpha_{\mu i}$. The validity of this assumption is evident in the case of the leading exponent $\alpha_{\nu 0} \equiv d_c$, but not so clear for the other exponents. However, if we accept the assumption, it can be verified later using the numerical results. Thus, if we assume that $\alpha_{\nu i} = \alpha_{\mu i}$, one can construct such a linear combination of the three quantities l_i, for which two terms (out of the three leading ones) cancel out. It suffices to find a root-mean-square fit by minimizing the expression

$$S(d) =$$
$$\min_{A_1, A_2, A_3} \sum_{i=i_0}^{i_{max}} \left[\frac{N^{d}(i) - A_1 l_1(i) - A_2 l_2(i) - A_3 l_3(i)}{\sigma_i} \right]^2 .$$
(7)

Here, σ_i is the variance of the expression $A_1 l_1(i) - A_2 l_2(i) - A_3 l_3(i)$, and can be expressed via the individual variances of the quantities $l(i)$, and the pair-wise correlation coefficients; all this is easily calculated, using the Monte-Carlo simulation results. The resulting curves $\log_{10}[S(d)/(n-4)]$ are given in Fig. 4; here n is the number of terms in the sum $S(d)$ (so that $n-4$ is the number of degrees of freedom). Note that the two smallest data-points (with $N_0 = 128$ and $N_1 = 196$) have been excluded from the sum, otherwise, the residual squares would have been unacceptably large [resulting in $S/(n-4) \gtrsim 1$]. Evidently, for $N \leq 196$, the higher ($\nu \geq 3$) terms of

FIG. 3: Finite-size effects for $H = 0$ give rise to a very slow convergence of the estimated fractal dimension $\tilde{d}_c(N) = \ln[l_i(N_{k+1}/l_i(N_k)]/\ln(N_{k+1}/N_k)$, $N = \sqrt{N_k N_{k+1}}$.

the expansion (6) become important. The existence of three sharp minima for all the curves in Fig. 4 confirms the assumption which was made about the equality of the leading exponents. Indeed, if the second and third asymptotic expansion terms of l_1, l_2, and l_3 were linearly independent, they would not cancel out in Eq. (7), and there would be no minima with $S/(n-4) \sim 1$.

This novel method of interpreting the finite-size simulation results allowed us to calculate not only the fractal dimension of the "oceanic coastlines" (see Fig. 5), but also the second and third exponents in the asymptotic expansion of the length of "oceanic coastlines" (Fig. 6).
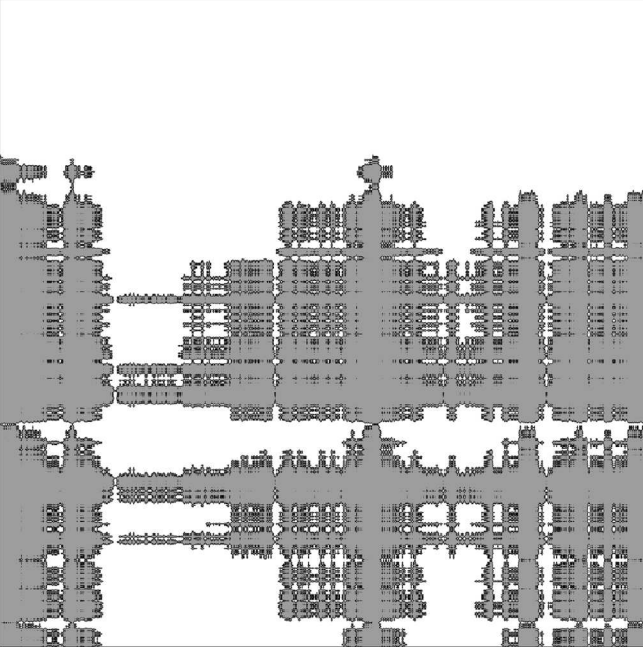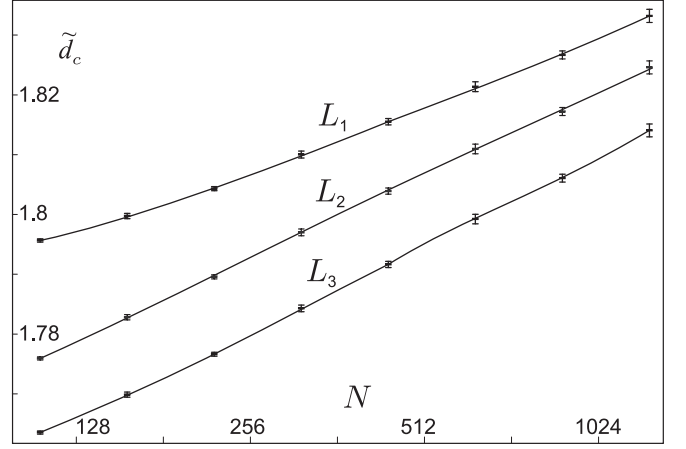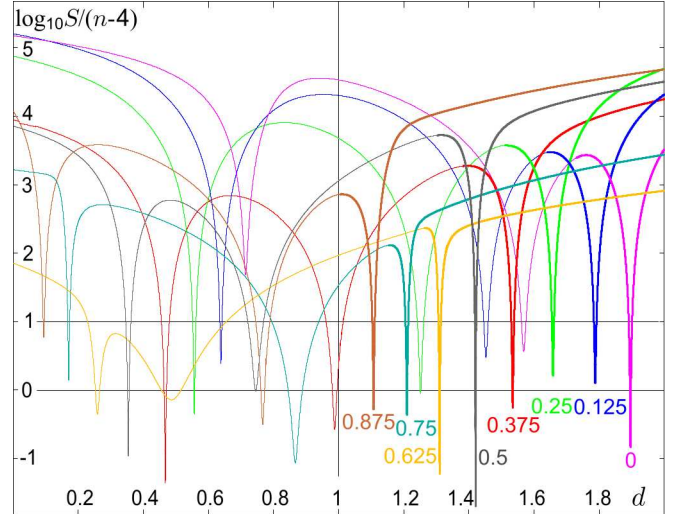

FIG. 2: The polygon ($N = 2048$, $H = 0.5$) has been flooded so that the flooded region (grey area) connects left and right edges of the polygon. The striped patterns (in horizontal and vertical directions) are due to the (1+1)D geometry of the surface.

FIG. 4: The quantity $\log_{10}[S(d)/(n-4)]$ [see Eq. (7)] is plotted versus d for different values of the Hurst exponent H (each curve is labeled with a number indicating the value of H). For all the curves, three minima are clearly distinct. At these minima, $d = \alpha_i$, where α_i is the i-th leading exponent of the asymptotic expansion of the quantities l_1, l_2, and l_3.

The values $d_c = 1.8965 \pm 0.0025$ and $\alpha_1 = 1.58 \pm 0.01$ for $H = 0$ are relatively close to each other, exactly as it was expected, judging by the finite-size scaling in Fig. 3.

Note that for $H \leq -3/4$ (uncorrelated percolation), there is analytic result $d_c = 91/48 \approx 1.8958$ [14], which is indistinguishable (at our numerical uncertainty) from our result for $H = 0$. While the author is not able to give a rigorous proof, it seems natural to assume that the function d_f is a monotonically decreasing function of H. Indeed, a larger H means a stronger influence of long scales, which results in less twisted isolines, and a lesser number of small islands (note that these arguments do not apply for d_u: excessive twists of the hull can efficiently screen and smooth the perimeter). This leads us to the conjecture $d_c \equiv 91/48$ for $H \leq 0$.

In conclusion, we have introduced the concept of "oceanic coastlines", and shown that they are the extension of the concept of percolation clusters. The finding that their fractal dimension d_c is clearly distinct from both the dimension of single isolines d_h and the dimension of the full set of isolines at a given height d_f is partic-

FIG. 6: Second and third largest exponents (α_1 and α_2) of the asymptotic expansion of the "oceanic coastline" length are plotted versus the Hurst exponent H.

ularly important, because in the context of Earth's landscapes, the coastlines of "oceanic islands" have been often mistakenly treated as the full set of isolines [8]. Based on the numerical results, we have conjectured that for $H \leq 0$, the fractal dimension of percolation clusters is super-universal, $d_c \equiv 91/48$. Last but not least, a novel and universal approach to the problem of interpreting the finite-size simulation results has been devised.
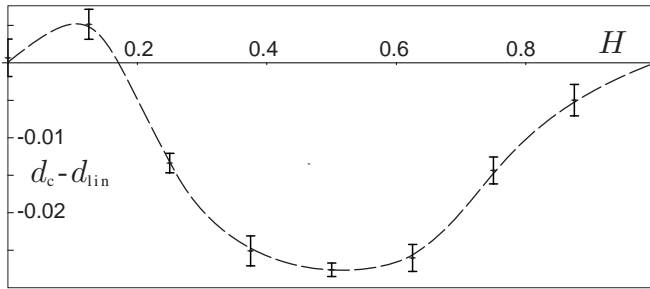

FIG. 5: The mismatch between a linear estimate $d_{lin} = \frac{91}{48} - H\frac{43}{48}$ and the fractal dimension of the "oceanic coastline" d_c is plotted versus the Hurst exponent H.
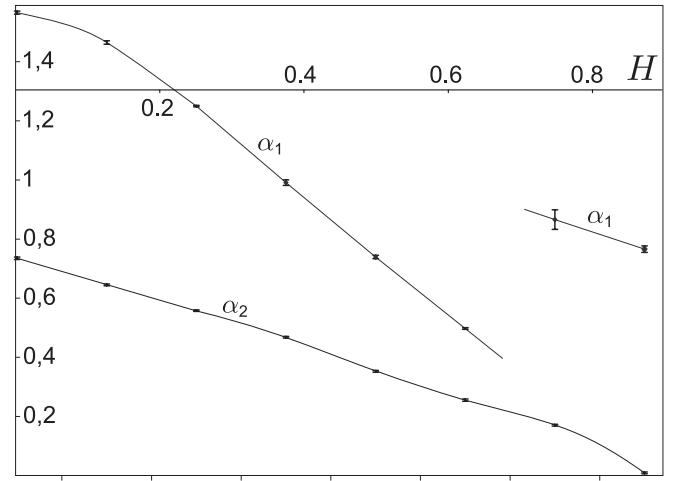
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